Classification of integrable hydrodynamic chains

A.V. Odesskii [1,2], **V.V. Sokolov** [1]

[1] L.D. Landau Institute for Theoretical Physics (Russia)

[2] Brock University (Canada)

Abstract Using the method of hydrodynamic reductions, we find all integrable infinite (1+1)-dimensional hydrodynamic-type chains of shift one. A class of integrable infinite (2+1)-dimensional hydrodynamic-type chains is constructed.

MSC numbers: 17B80, 17B63, 32L81, 14H70

Address: L.D. Landau Institute for Theoretical Physics of Russian Academy of Sciences, Kosygina 2, 119334, Moscow, Russia

E-mail: aodesski@brocku.ca, sokolov@itp.ac.ru

Contents

1 Introduction

We consider integrable infinite quasilinear chains of the form

$$u_{\alpha,t} = \phi_{\alpha,1} u_{1,x} + \cdots + \phi_{\alpha,\alpha+1} u_{\alpha+1,x}, \qquad \alpha = 1, 2, ..., \qquad \phi_{\alpha,\alpha+1} \neq 0, \tag{1.1}$$

where $\phi_{\alpha,j} = \phi_{\alpha,j}(u_1, ..., u_{\alpha+1})$. Two chains are called *equivalent* if they are related by a transformation of the form

$$u_\alpha \to \Psi_\alpha(u_1, ..., u_\alpha), \qquad \frac{\partial \Psi_\alpha}{\partial u_\alpha} \neq 0, \qquad \alpha = 1, 2, ... \tag{1.2}$$

By integrability we mean the existence of an infinite set of hydrodynamic reductions [1, 2, 3, 4, 5, 6].

Example 1. The Benney equations [7, 8, 9]

$$u_{1,t} = u_{2,x}, \qquad u_{2,t} = u_1 u_{1,x} + u_{3,x}, \ldots \quad u_{\alpha t} = (\alpha - 1) u_{\alpha-1} u_{1,x} + u_{\alpha+1,x}, \ldots \tag{1.3}$$

provide the most known example of integrable chain (1.1). The hydrodynamic reductions for the Benney chain were investigated in [10]. □

In [4, 5, 6] integrable divergent chains of the form

$$u_{1t} = F_1(u_1, u_2)_x, \quad u_{2t} = F_2(u_1, u_2, u_3)_x, \cdots, \quad u_{it} = F_i(u_1, u_2, ..., u_{i+1})_x, \cdots \tag{1.4}$$

were considered. In [6] some necessary integrability conditions were obtained. Namely, a nonlinear overdetermined system of PDEs for functions F_1, F_2 was presented. The general solution of the system was not found. Another open problem was to prove that the conditions are sufficient. In other words, for any solution F_1, F_2 of the system one should find functions $F_i, i > 2$ such that the resulting chain is integrable.

Probably any integrable chain (1.1) is equivalent to a divergent chain. However, the divergent coordinates are not suitable for explicit formulas. Our main observation is that a convenient coordinates are those, in which the so-called Gibbons-Tsarev type system (GT-system) related to integrable chain is in a canonical form.

Using our version (see [11, 12]) of the hydrodynamic reduction method, we describe all integrable chains (1.1). We establish an one-to-one correspondence between integrable chains (1.1) and infinite triangular GT-systems of the form

$$\partial_i p_j = \frac{P(p_i, p_j)}{p_i - p_j} \partial_i u_1, \qquad i \neq j, \tag{1.5}$$

$$\partial_i \partial_j u_1 = \frac{Q(p_i, p_j)}{(p_i - p_j)^2} \partial_i u_1 \partial_j u_1, \qquad i \neq j, \tag{1.6}$$

$$\partial_i u_m = (g_{m,0} + g_{m,1} p_i + \cdots + g_{m,m-1} p_i^{m-1}) \partial_i u_1, \qquad g_{m,j} = g_{m,j}(u_1, ..., u_m), \quad g_{m,m-1} \neq 0,$$

where $m = 2, 3, ...$ and $i, j = 1, 2, 3$. The functions P, Q are polynomials quadratic in each of variables p_i and p_j, with coefficients being functions of u_1, u_2. The functions $p_1, p_2, p_3,\ u_1, u_2, ...$ in (3.11) depend on r^1, r^2, r^3, and $\partial_i = \frac{\partial}{\partial r^i}$.

Example 1-1 (continuation of Example 1.) The system (1.5),(1.6) corresponding to the Benney chain has the following form

$$\partial_i p_j = \frac{\partial_i u_1}{p_i - p_j}, \qquad \partial_i \partial_j u_1 = \frac{2\, \partial_i u_1\, \partial_j u_1}{(p_i - p_j)^2}, \tag{1.7}$$

$$\partial_i u_m = (-(m-2)u_{m-2} - \cdots - 2u_2 p_i^{m-2} - u_1 p_i^{m-3} + p_i^{m-1})\, \partial_i u_1. \tag{1.8}$$

Equations (1.7) were firstly obtained in [10]. □

Given GT-system (1.5), (1.6) the coefficients of (1.1) are uniquely defined by the following relations

$$p_i \partial_i u_m = \phi_{m,1} \partial_i u_1 + \cdots + \phi_{m,m+1} \partial_i u_{m+1}, \qquad m = 2, 3, ... \tag{1.9}$$

Namely, equating the coefficients at different powers of p_i in (1.9), we get a triangular system of linear algebraic equations for $\phi_{i,j}$. Thus, the classification problem for chains (1.1) is reduced to a description of all GT-systems (1.5), (1.6) . The latter problem is solved in Section 4-6.

The paper is organized as follows. Following [11, 12], we recall main definitions in Section 2 (see [1, 2, 3, 11] for details). We consider only 3-component hydrodynamic reductions since the existence of reductions with $N > 3$ gives nothing new [1]. In Section 3 we formulate our previous results that are needed in the paper. Section 4 is devoted to a classification of admissible polynomials P and Q in (1.5), (1.6). In Sections 5,6 we construct integrable chains for the generic case and for some degenerations. Section 6 also contains examples of (2+1)-dimensional infinite hydrodynamic-type chains integrable from the viewpoint of the method of hydrodynamic reductions. Infinitesimal symmetries of GT-systems are studied in Section 7. These symmetries seem to be important basic objects in the hydrodynamic reduction approach.

Acknowledgments. Authors thank M.V. Pavlov for fruitful discussions. V.S. is grateful to Brock University for hospitality. He was partially supported by the RFBR grants 08-01-464, 09-01-22442-KE, and NS 3472.2008.2.

2 Integrable chains and hydrodynamic reductions

According to [1, 2, 3, 4, 5, 6] a chain (1.1) is called *integrable* if it admits sufficiently many so-called hydrodynamic reductions.

Definition. A hydrodynamic (1+1)-dimensional N-component reduction of a chain (1.1) is a semi-Hamiltonian (see formula (3.18)) system of the form

$$r^i_t = p_i(r^1, ..., r^N) r^i_x, \qquad i = 1, .., N \tag{2.10}$$

and functions $u_j(r^1,...,r^N),\ j=1,2,...$ such that for each solution of (2.10) functions $u_j = u_j(r^1,...,r^N),\ i=1,...$ satisfy (1.1).

Substituting $u_i = u_i(r^1,...,r^N),\ i=1,...$ into (1.1), calculating t and x-derivatives by virtue of (2.10) and equating coefficients at r^s_x to zero, we obtain

$$\partial_s u_\alpha p_s = \phi_{\alpha,1}\partial_s u_1 + \cdots + \phi_{\alpha,\alpha+1}\partial_s u_{\alpha+1}, \qquad \alpha = 1,2,...$$

It is clear from this system that

$$\partial_s u_k = g_k(p_s,u_1,...,u_k)\,\partial_s u_1, \qquad k=2,3,...$$

where $g_k(p,u_1,...,u_k)$ is a polynomial of degree $k-1$ in p for each $k=2,3,...$ Compatibility conditions $\partial_i\partial_j u_k = \partial_j\partial_i u_k$ give us a system of linear equations for $\partial_i p_j,\ \partial_j p_i,\ \partial_i\partial_j u_1,\ i\neq j$. This system should have a solution (otherwise we would not have sufficiently many reductions). Moreover, expressions for $\partial_s u_k,\ k=2,3,...,\ \partial_j p_i,\ \partial_i\partial_j u_1,\ i\neq j$ should be compatible and form a so-called GT-system.

Remark. In the sequel we assume $N=3$ because the case $N>3$ gives nothing new [1].

3 GT-systems

Definition. A compatible system of PDEs of the form

$$\begin{gathered}\partial_i p_j = f(p_i,p_j,u_1,...,u_n),\ \partial_i u_1 \qquad j\neq i,\\ \partial_i\partial_j u_1 = h(p_i,p_j,u_1,...,u_n)\,\partial_i u_1\partial_j u_1, \qquad j\neq i,\\ \partial_i u_k = g_k(p_i,u_1,...,u_n)\,\partial_i u_1, \qquad k=1,...,n-1,\end{gathered} \tag{3.11}$$

where $i,j=1,2,3$ is called *n-fields GT-system*. Here $p_1,p_2,p_3,\ u_1,...,u_n$ are functions of r^1,r^2,r^3 and $\partial_i = \frac{\partial}{\partial r^i}$.

Definition. Two GT-systems are called *equivalent* if they are related by a transformation of the form

$$p_i \to \lambda(p_i,u_1,...,u_n), \tag{3.12}$$

$$u_k \to \mu_k(u_1,...,u_n), \qquad k=1,...,n. \tag{3.13}$$

Example 2 [13]. Let a_0,a_1,a_2 be arbitrary constants, $R(x) = a_2x^2 + a_1x + a_0$. Then the system

$$\partial_i p_j = \frac{a_2p_j^2 + a_1p_j + a_0}{p_i - p_j}\partial_i u_1, \qquad \partial_i\partial_j u_1 = \frac{2a_2p_ip_j + a_1(p_i+p_j) + 2a_0}{(p_i-p_j)^2}\partial_i u_1\partial_j u_1 \tag{3.14}$$

is an one-field GT-system. The original Gibbons-Tsarev system (1.7) corresponds to $a_2 = a_1 = 0, a_0 = 1$. The polynomial $R(x)$ can be reduced to one of the following canonical forms: $R=1$,

$R = x$, $R = x^2$, or $R = x(x-1)$ by a linear transformation (3.12). A wide class of integrable 3D-systems of hydrodynamic type related to (3.14) is described in [13]. An elliptic version of this GT-system and the corresponding integrable 3D-systems were constructed in [15]. □

Definition. An additional system

$$\partial_i u_k = g_k(p_i, u_1, ..., u_{n+m})\partial_i u_n, \qquad k = n+1, ..., n+m \tag{3.15}$$

such that (3.11) and (3.15) are compatible is called *an extension* of (3.11) by fields $u_{n+1}, ..., u_{n+m}$.

It turns our that

$$\partial_i u_{n+1} = f(p_i, u_{n+1}, u_1, ..., u_n)\, \partial_i u_1$$

is an extension for GT-system (3.11). Stress that here f is the same function as in (3.11). We call this extension *the regular extension* by u_{n+1}.

Example 2-1. The generic case of Example 2 corresponds to $R = x(x-1)$. The regular extension by u_2 is given by

$$\partial_i u_2 = \frac{u_2(u_2-1)}{p_i - u_2}\partial_i u_1.$$

If we express u_1 from this formula and substitute it to (3.14), we get the following one-field GT-system

$$\begin{aligned} \partial_i p_j &= \frac{p_j(p_j-1)(p_i-u_1)}{u_1(u_1-1)(p_i-p_j)}\partial_i u_1, \\ \partial_i\partial_j u_1 &= \frac{p_ip_j(p_i+p_j) - p_i^2 - p_j^2 + (p_i^2 + p_j^2 - 4p_ip_j + p_i + p_j)u_1}{u_1(u_1-1)(p_i-p_j)^2}\partial_i u_1 \partial_j u_1. \qquad \square \end{aligned} \tag{3.16}$$

The second basic notion of the hydrodynamic reduction method is so-called GT-family of (1+1)-dimensional hydrodynamic-type systems.

Definition. An (1+1)-dimensional 3-component hydrodynamic-type system of the form

$$r^i_t = v^i(r^1, ..., r^N)\, r^i_x, \qquad i = 1, 2, 3, \tag{3.17}$$

is called semi-Hamiltonian if the following relation holds

$$\partial_j \frac{\partial_i v^k}{v^i - v^k} = \partial_i \frac{\partial_j v^k}{v^j - v^k}, \qquad i \neq j \neq k. \tag{3.18}$$

Definition. A Gibbons-Tsarev family associated with the Gibbons-Tsarev type system (4.25) is a (1+1)-dimensional hydrodynamic-type system of the form

$$r^i_t = F(p_i, u_1, ..., u_m) r^i_x, \qquad i = 1, 2, 3, \tag{3.19}$$

semi-Hamiltonian by virtue of (3.11).

Example 2-2 [13]. Applying the regular extension to the generic GT-system (3.14) two times, we get the following GT-system:

$$\partial_i p_j = \frac{p_j(p_j-1)}{p_i-p_j}\partial_i w, \qquad \partial_{ij} w = \frac{2p_ip_j-p_i-p_j}{(p_i-p_j)^2}\partial_i w \partial_j w, \qquad i \neq j, \tag{3.20}$$

$$\partial_i u_j = \frac{u_j(u_j-1)\partial_i w}{p_i-u_j}, \qquad j=1,2. \tag{3.21}$$

Consider the generalized hypergeometric [14] linear system of the form

$$\frac{\partial^2 h}{\partial u_j \partial u_k} = \frac{s_j}{u_j-u_k}\cdot\frac{\partial h}{\partial u_k} + \frac{s_k}{u_k-u_j}\cdot\frac{\partial h}{\partial u_j}, \qquad j \neq k, \tag{3.22}$$

$$\begin{aligned}\frac{\partial^2 h}{\partial u_j \partial u_j} = -\left(1+\sum_{k=1}^{n+2} s_k\right)\frac{s_j}{u_j(u_j-1)}\cdot h + \frac{s_j}{u_j(u_j-1)}\sum_{k\neq j}^{n}\frac{u_k(u_k-1)}{u_k-u_j}\cdot\frac{\partial h}{\partial u_k}+ \\ \left(\sum_{k\neq j}^{n}\frac{s_k}{u_j-u_k}+\frac{s_j+s_{n+1}}{u_j}+\frac{s_j+s_{n+2}}{u_j-1}\right)\cdot\frac{\partial h}{\partial u_j}.\end{aligned} \tag{3.23}$$

Here $i,j=1,2$ and $s_1,...,s_4$ are arbitrary parameters. It easy to verify that this system is in involution and therefore the solution space is 3-dimensional. Let h_1,h_2,h_3 be a basis of this space. For any h we put

$$S(p,h) = u_1(u_1-1)(p-u_2)\frac{hh_{1,u_1}-h_{u_1}h_1}{h_1} + u_2(u_2-1)(p-u_1)\frac{hh_{1,u_2}-h_{u_2}h_1}{h_1}.$$

Then the formula

$$F = \frac{S(p,h_3)}{S(p,h_2)} \tag{3.24}$$

defines the generic linear fractional GT-family for (3.20). □

4 Canonical forms of GT-systems associated with integrable chains

For integrable chains the corresponding GT-systems involve infinite number of fields $u_i,\ i = 1,2,...$ (see Example 1-1). In this Section we show that these GT-systems are equivalent to infinite triangular extensions of one-field GT-systems from Examples 2,3.

A compatible system of PDEs of the form

$$\begin{aligned}\partial_i p_j = f(p_i,p_j,u_1,...,u_n)\,\partial_i u_1 \qquad &,\ i\neq j,\\ \partial_i u_k = g_k(p_i,u_1,...,u_k)\,\partial_i u_1, \qquad &k=1,2,...,,\end{aligned} \tag{4.25}$$

$$\partial_i\partial_j u_1 = h(p_i, p_j, u_1, ..., u_n)\, \partial_i u_1 \partial_j u_1, \qquad i \neq j,$$

where $i, j = 1, 2, 3$ is called *triangular GT-system.* Here $p_1, p_2, p_3,\ u_1, u_2, ...$ are functions of r^1, r^2, r^3, and $\partial_i = \frac{\partial}{\partial r^i}$.

Definition. A chain (1.1) is called integrable if there exists a Gibbons-Tsarev type system of the form (4.25) and a Gibbons-Tsarev family

$$r^i_t = F(p_i, u_1, ..., u_m) r^i_x, \qquad i = 1, 2, 3, \tag{4.26}$$

such that (1.1) holds by virtue of (4.25), (4.26).

Due to the equivalence transformations (3.12) we can assume without loss of generality that

$$F(p, u_1, ..., u_m) = p. \tag{4.27}$$

Under this assumption we have

$$u_{j,t} = \sum_s \partial_s u_j r^s_t = \sum_s \partial_s u_j p_s r^s_x.$$

and similar

$$u_{j,x} = \sum_s \partial_s u_j r^s_x.$$

Substituting these expressions into (1.1) and equating coefficients at r^s_x to zero, we obtain

$$\partial_s u_\alpha p_s = \phi_{\alpha,1} \partial_s u_1 + \cdots + \phi_{\alpha,\alpha+1} \partial_s u_{\alpha+1}, \qquad \alpha = 1, 2, ...$$

Using (4.25) and replacing p_s by p, we get

$$p = \phi_{1,1} + \phi_{1,2} g_2,\ \ p g_2 = \phi_{2,1} + \phi_{2,2} g_2 + \phi_{2,3} g_3,\ \ p g_3 = \phi_{3,1} + \phi_{3,2} g_2 + \phi_{3,3} g_3 + \phi_{3,4} g_4, ...$$

Solving this system with respect to $g_2,\ g_3, ...$, we obtain

$$g_i(p) = \psi_{i,0} + \psi_{i,1} p + ... + \psi_{i,i-1} p^{i-1}.$$

Here $\psi_{i,j}$ are functions of $u_1, ..., u_i$. For example,

$$g_2 = -\frac{p}{\phi_{1,2}} - \frac{\phi_{1,1}}{\phi_{1,2}}. \tag{4.28}$$

Remark. Since we assume that $\phi_{i,i-1} \neq 0$, we have $\psi_{i,i-1} \neq 0$ for all i. Therefore $g_1 = 1, g_2, ...$ is a basis in the linear space of all polynomials in p. The coefficients $\phi_{i,j}$ of our chain are just entries of the matrix of multiplication by p in this basis. More generally, if we don't normalize $F = p$, then the coefficients $\phi_{i,j}$ can be found from the equations

$$\begin{aligned} &F(p) = \phi_{1,1} + \phi_{1,2} g_2, \qquad F(p) g_2 = \phi_{2,1} + \phi_{2,2} g_2 + \phi_{2,3} g_3, \\ &F(p) g_3 = \phi_{3,1} + \phi_{3,2} g_2 + \phi_{3,3} g_3 + \phi_{3,4} g_4, ... \end{aligned} \tag{4.29}$$

Compatibility conditions $\partial_i\partial_j u_\alpha = \partial_j\partial_i u_\alpha,\ \alpha = 2,3,4$ give a system of linear equations for $\partial_i p_j,\ \partial_j p_i,\ \partial_i\partial_j u_1$. Solving this system, we obtain formulas (1.5),(1.6), where in principal $P,\ Q$ could depend on u_1, u_2, u_3, u_4. However, it follows from compatibility conditions $\partial_i\partial_j p_k = \partial_j\partial_i p_k$ that $P,\ Q$ depend on $u_1,\ u_2$ only.

Written (1.5) in the form

$$\partial_i p_j = \left(\frac{R(p_j)}{p_i - p_j} + (z_4 p_j^2 + z_5 p_j + z_6)p_i + z_4 p_j^3 + z_3 p_j^2 + z_7 p_j + z_8\right)\partial_i u_1, \tag{4.30}$$

where $R(x) = z_4x^4 + z_3x^3 + z_2x^2 + z_1x + z_0$, one can derive from the compatibility conditions $\partial_i\partial_j p_k = \partial_j\partial_i p_k$, $\partial_i\partial_j u_1 = \partial_j\partial_i u_1$ that the equation (1.6) has the following form

$$\partial_i\partial_j u_1 = \left(\frac{2z_4p_i^2p_j^2 + z_3p_ip_j(p_i+p_j) + z_2(p_i^2+p_j^2) + z_1(p_i+p_j) + 2z_0}{(p_i-p_j)^2} + z_9\right)\partial_i u_1\partial_j u_1. \tag{4.31}$$

It is easy to verify that we can normalize $z_9 = z_6 - z_7,\ g_2 = p$ by a transformation (1.2). Then the coefficients $z_i(x,y), i = 0, ..., 8$ satisfy the following pair of compatible dynamical systems with respect to y and x:

$$z_{0,y} = 2z_0z_5 - z_1z_6, \qquad z_{1,y} = 4z_0z_4 + z_1z_5 - 2z_2z_6, \qquad z_{2,y} = 3z_1z_4 - 3z_3z_6,$$
$$z_{3,y} = 2z_2z_4 - z_3z_5 - 4z_4z_6, \qquad z_{4,y} = z_3z_4 - 2z_4z_5, \qquad z_{5,y} = z_4z_7 - z_4z_6 - z_5^2,$$
$$z_{6,y} = z_4z_8 - z_5z_6, \qquad z_{7,y} = 2z_1z_4 - 2z_3z_6 - z_5z_6 + z_4z_8, \qquad z_{8,y} = 2z_0z_4 - z_6^2 - z_6z_7 + z_5z_8,$$

and

$$z_{0,x} = -z_0z_2 - z_0z_6 + 3z_0z_7 - z_1z_8, \qquad z_{1,x} = -z_1z_2 + 3z_0z_3 - z_1z_6 + 2z_1z_7 - 2z_2z_8,$$
$$z_{2,x} = -z_2^2 + 2z_1z_3 + 4z_0z_4 - z_2z_6 + z_2z_7 - 3z_3z_8, \qquad z_{3,x} = 3z_1z_4 - z_3z_6 - 4z_4z_8,$$
$$z_{4,x} = z_2z_4 - z_4z_6 - z_4z_7, \qquad z_{5,x} = z_1z_4 - z_5z_6 - z_4z_8, \qquad z_{6,x} = z_0z_4 - z_6^2,$$
$$z_{7,x} = z_1z_3 + 3z_0z_4 + z_1z_5 - z_2z_6 - z_2z_7 + z_7^2 - z_3z_8 - 2z_5z_8,$$
$$z_{8,x} = z_0z_3 + z_0z_5 - z_2z_8 - 2z_6z_8 + z_7z_8.$$

These is a complete description of the GT-systems related to integrable chains (1.1).

To solve the dynamical systems we bring the polynomial R to a canonical form sacrificing to the normalization (4.27).

It is obvious that linear transformations $p_i \to ap_i + b$, where a, b are functions of u_1, u_2, preserve the form of GT-system (4.30),(4.31). Moreover, there exist transformations of the form

$$p_i = \frac{a\bar{p}_i + b}{\bar{p}_i - \psi}, \qquad i = 1,2,3 \tag{4.32}$$

preserving the form of GT-system (4.30),(4.31). Such admissible transformations are described by the following conditions:

$$a_{u_2} = z_4(b + a\psi), \qquad b_{u_2} = z_4 b\psi + z_5 b - z_6 a, \qquad \psi_{u_2} = z_4\psi^2 + z_5\psi + z_6.$$

Under transformations (4.32) the polynomial R is transformed by the following simple way:

$$R(p_i) \to (p_i - \psi)^4 R\Big(\frac{ap_i + b}{p_i - \psi}\Big).$$

Suppose that R has distinct roots. It is possible to verify that by an admissible transformation (4.32) we can move three of the four roots to 0, 1 and ∞. It follows from compatibility conditions for the GT-system that then the fourth root $\lambda(u_1, u_2)$ does not depend on u_2. Making transformation of the form $u_1 \to q(u_1)$ we arrive at the canonical forms $\lambda = u_1$ or $\lambda = const$. It is straightforwardly verified that in the first case equations (4.30), (4.31) coincides with (3.16). In the second case the GT-system does not exist.

In the case of multiple roots the polynomial $R(x)$ can be reduced to one of the following forms: $R = 0$, $R = 1$, $R = x$, $R = x^2$, or $R = x(x-1)$. In all these cases equations (4.30), (4.31) coincides with the corresponding equations from Example 2.

Thus, the following statement is valid:

Proposition 1. There are 6 non-equivalent cases of GT-systems (4.30), (4.31). The canonical forms are:

Case 1 : (3.16) (generic case);
Case 2 : (3.14) with $R(x) = x(x-1)$;
Case 3 : (3.14) with $R(x) = x^2$;
Case 4 : (3.14) with $R(x) = x$;
Case 5 : (3.14) with $R(x) = 1$.
Case 6 : (3.14) with $R(x) = 0$. □

Remark. Cases 2-6 can be obtained from Case 1 by appropriate limit procedures. For example, Case 2 corresponds to the limit $u_1 \to \frac{u_1}{\varepsilon}$, $\varepsilon \to 0$.

It follows from (4.27), (4.28) that for any canonical form the functions F and g_2 have the following structure:

$$g_2(p_i) = \frac{k_1 p_i + k_2}{k_3 p_i + k_4}, \qquad F(p_i) = \frac{f_1 p_i + f_2}{k_3 p_i + k_4}, \tag{4.33}$$

where the coefficients are functions of u_1, u_2.

Lemma 1. For the Case 1 any function g_2 can be reduced by an appropriate transformation

$\bar{u}_2 = \sigma(u_1, u_2)$ to one of the following canonical forms:

$$\mathbf{a_1}: \quad g_2(p) = \frac{u_2(u_2-1)(p-u_1)}{u_1(u_1-1)(p-u_2)} \qquad \text{(regular extension);}$$

$$\mathbf{b_1}: \quad g_2(p) = \frac{1}{p-u_1};$$

$$\mathbf{c_1}: \quad g_2(p) = \frac{u_1^{-\lambda}(u_1-1)^{\lambda-1}}{p-\lambda} \qquad \lambda = 1, 0;$$

$$\mathbf{d_1}: \quad g_2(p) = \frac{u_1-u_2}{u_1(u_1-1)}p + \frac{u_2-1}{u_1-1}. \quad \square$$

The GT-system from the Case 1 possesses a discrete automorphism group S_4 interchanging the points $0, 1, \infty, u_1$. The group is defined by generators

$$\sigma_1 : u_1 \to 1-u_1, \quad p_i \to 1-p_i, \qquad \sigma_2 : u_1 \to \frac{u_1}{u_1-1}, \quad p_i \to \frac{p_i}{p_i-1},$$

and

$$\sigma_3 : u_1 \to 1-u_1, \quad p_i \to \frac{(1-u_1)p_i}{p_i-u_1}.$$

Up to this group the cases b_1, c_1, d_1 are equivalent and one can take say the case d_1 for further consideration. The case a_1 is invariant with respect to the group.

Remark. The cases b_1, c_1, d_1 are degenerations of the case a_1. Namely, they can be obtained as appropriate limit $u_2 \to u_1$, $u_2 \to \lambda$, $u_2 \to \infty$ correspondingly.

All possible functions g_2 for Cases 2-5 are described in the following

Lemma 2. For the GT-system (3.14) (excluding Case 6) any function g_2 can be reduced by an appropriate transformation $\bar{u}_2 = \sigma(u_1, u_2)$ to one of the following canonical forms:

$$\mathbf{a_2}: \quad g_2(p) = \frac{R(u_2)}{p-u_2} \qquad \text{(regular extension);}$$

$$\mathbf{b_2}: \quad g_2(p) = \frac{1}{p-\lambda}, \quad \text{where} \quad R(\lambda) = 0;$$

$$\mathbf{c_2}: \quad g_2(p) = p - a_2 u_2.$$

The discrete automorphism of the GT-system interchanges the roots of R in the case b_2. $\square$

Lemma 3. For the GT-system (3.14) with $R(x) = 0$ (Case 6) any function g_2 can be reduced to $g_2(p) = p$ by an appropriate transformation $\bar{u}_2 = \sigma(u_1, u_2)$. Furthermore, the corresponding triangular GT-system has the form

$$\partial_i p_j = 0, \qquad \partial_i \partial_j u_1 = 0, \qquad \partial_i u_k = p_i^{k-1} u_1, \quad k = 2, 3, ... \quad \square \tag{4.34}$$

5 Generic case

The next step in the classification is to find all functions F of the form (4.28) for each pair consisting of a GT-system from Proposition 1 and the corresponding g_2 from Lemmas 1-3. The semi-Hamiltonian condition (3.18) yields a non-linear system of PDEs for the functions $f_1(u_1, u_2)$, $f_2(u_1, u_2)$. For each case this system can be reduced to the linear generalized hypergeometric system (3.22), (3.23) with a special set of parameters s_1, s_2, s_3, s_4 or to a degeneration of this system.

The general linear fractional GT-family for the generic case 1, $\mathbf{a_1}$ is given by (3.24). According to (4.33), the additional restriction is that the root of the denominator has to be equal u_2. It is easy to verify that this is equivalent to $s_2 = 0, h_{1,u_2} = h_{2,u_2} = 0$. The latter means that $h_1(u_1), h_2(u_1)$ are linear independent solutions of the standard hypergeometric equation

$$u(u-1)\,h(u)'' + [s_1 + s_3 - (s_3 + s_4 + 2s_1)\,u]\,h(u)' + s_1(s_1 + s_3 + s_4 + 1)\,h(u) = 0. \qquad (5.35)$$

The function $h_3(u_1, u_2)$ is arbitrary solution of (3.22), (3.23) with $s_2 = 0$ linearly independent of $h_1(u_1), h_2(u_1)$. Without loss of generality we can choose

$$h_3(u_1, u_2) = \int_0^{u_2} (t - u_1)^{s_1} t^{s_3} (t-1)^{s_4} dt.$$

Formula (3.24) gives

$$F(p, u_1, u_2) = \frac{f_1(u_1, u_2)\,p - f_2(u_1, u_2)}{p - u_2}, \qquad (5.36)$$

where

$$f_1 = \frac{u_2(u_2-1)h_1 h_{3,u_2} + u_1(u_1-1)(h_1 h_{3,u_1} - h_3 h_1')}{u_1(u_1-1)(h_1 h_2' - h_2 h_1')},$$

$$f_2 = \frac{u_1 u_2(u_2-1)h_1 h_{3,u_2} + u_2 u_1(u_1-1)(h_1 h_{3,u_1} - h_3 h_1')}{u_1(u_1-1)(h_1 h_2' - h_2 h_1')}.$$

Notice that $h_1 h_2' - h_2 h_1' = const(u_1 - 1)^{s_1+s_4} u_1^{s_1+s_3}$.

For integer values of s_1, s_3, s_4 the hypergeometric system can be solved explicitly. For example, if $s_1 = s_3 = s_4 = 0$, the above formulas give rise to $F = g_2$. If $s_4 = -2 - s_1 - s_3$ then

$$F = \frac{(u_2 - u_1)^{s_1+1} u_2^{s_3+1} (u_2 - 1)^{-1-s_1-s_3}}{p - u_2};$$

if $s_4 = 0$, then

$$F = \frac{(p-1)(u_2 - u_1)^{s_1+1} u_2^{s_3+1} (u_1 - 1)^{-1-s_1}}{p - u_2}.$$

Now we are to find the functions $g_3, g_4, ...$ in (4.25). These functions are define up to arbitrary transformation (1.2), where $\alpha = 3, 4, ...$ In practice, one can look for functions $g_3, g_4, ...$ linear in $u_i, i > 2$ (cf. (1.8)). An extension linear in $u_i, i > 2$ is given by

$$g_3(p) = -\frac{(u_1 - u_2)(u_2 - 1)p}{u_1(u_1 - 1)(p - u_2)^2},$$

$$g_i(p) = \frac{(i-3)(u_1-u_2)(u_2-1)p\,u_i}{u_1(u_1-1)(p-u_2)^2} - \frac{(u_1-u_2)^{i-3}(u_2-1)^2 p(p-u_1)(p-1)^{i-4}}{u_1(u_1-1)^{i-2}(p-u_2)^{i-1}} - \sum_{s=1}^{i-4} \frac{(i-s-2)(u_1-u_2)^s(u_2-1)^2 p(p-u_1)(p-1)^{s-1}\,u_{i-s}}{u_1(u_1-1)^{s+1}(p-u_2)^{s+2}}.$$

The coefficients of the chain (1.1) corresponding to Case 1, $\mathbf{a_1}$ are determined from (4.29), where F is given by (5.36). Relations (4.29) are equivalent to a triangular system of linear algebraic equations. Solving this system, we find that for $i > 4$ coefficients of the chain read:

$$\phi_{i,i+1} = \frac{(u_1-1)(f_1u_2-f_2)}{(u_2-1)(u_1-u_2)} \stackrel{\text{def}}{=} Q_1, \qquad \phi_{i,i} = \frac{f_2-f_1}{u_2-1} \stackrel{\text{def}}{=} Q_2,$$

$$\phi_{i,4} = -u_iQ_1, \qquad \phi_{i,3} = -\Big((u_4+i-3)u_i + (2-i)u_{i+1}\Big)Q_1 \stackrel{\text{def}}{=} A_i,$$

and $\phi_{i,j} = 0$ for all remaining i, j. For $i \le 4$ we have

$$\begin{gathered} \phi_{1,1} = \frac{f_1u_1-f_2}{u_1-u_2}, \qquad \phi_{1,2} = -\frac{u_1}{u_2}Q_1, \\ \phi_{2,1} = \frac{(u_2-1)(f_1u_2-f_2)}{(u_1-1)(u_1-u_2)}, \qquad \phi_{2,2} = \frac{f_2u_1-f_1u_2^2}{u_2(u_1-u_2)}, \qquad \phi_{2,3} = f_1u_2-f_2, \\ \phi_{3,1} = \phi_{3,2} = 0, \qquad \phi_{3,3} = Q_2-(u_4-1)Q_1, \qquad \phi_{3,4} = -Q_1, \\ \phi_{4,1} = \phi_{4,2} = 0, \qquad \phi_{4,3} = A_4, \qquad \phi_{4,4} = Q_2-u_4Q_1, \qquad \phi_{4,5} = Q_1. \end{gathered} \tag{5.37}$$

The explicit formulas for other cases of Proposition 1 can be obtained by limits from the above formulas. We outline the limit procedures for the case 1, $\mathbf{d_1}$. In this case the limit is given by $u_2 \to u_1 + \varepsilon u_2, \quad \varepsilon \to 0$. It is easy to check that under this limit the extension a_1 turns to d_1. The limit of the system (3.22), (3.23) with $s_2 = 0$ can be easily found. The general solution of the system thus obtained is given by $h = c_1(u_2-u_1)^{1+s_1+s_3+s_4} + h_1$, where h_1 is the general solution of (5.35). Let h_1, h_2 be solutions of (5.35), and $h_3 = (u_2-u_1)^{1+s_1+s_3+s_4}$. Then the limit procedure in (5.36) gives rise to

$$F(p,u_1,u_2) = Q \times \Big((1+s_1+s_3+s_4)h_1(p-u_1) + u_1(u_1-1)h_1'\Big),$$

where

$$Q = (u_2-u_1)^{1+s_1+s_3+s_4}(u_1-1)^{-1-s_1-s_4}u_1^{-1-s_1-s_3}.$$

As usual, the most degenerate cases in classification of integrable PDEs could be interesting for applications. In our classification they are Case 5, c_2 and Case 6. The Benney chain (see Examples 1 and 1-1) belongs to Case 5, case c_2 (i.e $g_2 = p$). Any GT-family has the form $F = f_1(u_1,u_2)p + f_2(u_1,u_2)$. If $f_1 = 1$ then $F = p + k_2u_2 + k_1u_1$. The Benney case corresponds to

$k_1 = k_2 = 0$. For arbitrary k_i we get the Kupershmidt chain [16]. In the case $f_1 = A(u_1), A' \neq 0$ we obtain:

$$f_1 = k_2 \exp(\lambda u_1) + k_1, \qquad f_2 = k_2 k_3 \exp(\lambda u_1) + \lambda k_1 (k_3 u_1 - u_2).$$

In the generic case

$$F = \exp(\lambda u_2)(S_1(u_1)p + S_2(u_1)),$$

where the functions S_i can be expressed in terms of the Airy functions.

6 Trivial GT-system and $2{+}1$-dimensional integrable hydrodynamic chains

It was observed in [11] that (2+1)-dimensional systems of hydrodynamic type with the trivial GT-system usually admit some integrable multi-dimensional generalizations. For the chains such GT-system is defined by (4.34). That is why the Case 6 is of a great importance in our classification. The automorphisms of (4.34) are given by

$$p_j \to p_j,\ j = 1, ..., N,\ u_i \to \nu u_i + \gamma_i,\ i = 1, 2, ...; \tag{6.38}$$

$$p_j \to a p_j + b,\ j = 1, ..., N,\ u_i \to a^{i-1} u_i + (i-1) a^{i-2} b u_{i-2} + ... + b^{i-1} u_1, \quad i = 1, 2, ...$$

The corresponding GT-families are of the form $F(p) = A(u_1, u_2)p + B(u_1, u_2)$, where $A(x, y), B(x, y)$ satisfies the following system of PDEs:

$$\begin{gathered} AB_{yy} = A_y B_y, \quad AB_{xy} = A_y B_x, \quad AB_{xx} = A_x B_x, \\ AA_{yy} = A_y^2, \quad AA_{xy} = A_x A_y, \quad AA_{xx} = A_x^2 + A_x B_y - A_y B_x. \end{gathered} \tag{6.39}$$

This system can be easily solved in elementary functions. For each solution formula (4.29) defines the corresponding integrable chain (1.1).

It follows from (6.39) that there are two types of u_2-dependence:

1 (generic case). $\qquad F(p) = \exp(\lambda u_2)\Big(a(u_1)p + b(u_1)\Big),$

2. $\qquad F(p) = a(u_1)p + \lambda u_2 + b(u_1).$

In the first case there are two subcases: $b' \neq 0$ and $b' = 0$. The first subcase gives rise to

$$a = \sigma', \quad b = k_1 \sigma \qquad \sigma(x) = c_1 \exp(\mu_1 x) + c_2 \exp(\mu_2 x), \quad \text{where} \quad c_1 c_2 (\lambda k_1 - \mu_1 \mu_2) = 0.$$

The second subcase leads to

$$b = c_1, \qquad a(x) = c_2 \exp(\mu x) + c_3, \quad \text{where} \quad c_2 (c_1 \lambda - c_3 \mu) = 0.$$

The same subcases for the case 2 yield

$$a = \sigma', \quad b = k_1 \sigma \qquad \sigma(x) = c_1 + c_2 x + c_3 \exp(\mu x), \quad \text{where} \quad c_3 (\lambda - c_2 \mu) = 0,$$

and
$$b = c_1, \qquad a(x) = c_2 \exp(\mu x) + c_3, \quad \text{where} \quad c_2(\lambda - c_3\mu) = 0.$$

It is easy to verify that in the generic case the function F can be reduced by (6.38) to the form
$$F(p) = e^{u_2+u_1}(p-1) + e^{u_2-u_1}(p+1).$$
In this case the corresponding chain reads as
$$u_{k,t} = (e^{u_2+u_1} + e^{u_2-u_1})u_{k+1,x} + (e^{u_2-u_1} - e^{u_2+u_1})u_{k,x}, \qquad k = 1,2,3,... \tag{6.40}$$
As usual, this chain is the first member of an infinite hierarchy. The second flow of this hierarchy is given by
$$u_{k,\tau} = (e^{u_2+u_1} + e^{u_2-u_1})u_{k+2,x} + (u_3 - u_1)(e^{u_2+u_1} + e^{u_2-u_1})u_{k+1,x}+$$
$$(e^{u_2+u_1}(u_1 - u_3 - 1) + e^{u_2-u_1}(u_3 - u_1 - 1))u_{k,x}, \qquad k = 1,2,3,...$$

In the case 2 with $c_3 = \lambda = 0, k_1 = 1$ we get the chain
$$u_{k,t} = u_{k+1,x} + u_1 u_{k,x}, \qquad k = 1,2,3,... \tag{6.41}$$
This chain is equivalent to the chain of the so-called universal hierarchy [17]. The chain (6.41) is a degeneration of the chain
$$u_{k,t} = u_{k+1,x} + u_2 u_{k,x}, \qquad k = 1,2,3,... \tag{6.42}$$

Following the line of [3, 11] it is not difficult to find (2+1)-dimensional integrable generalizations for all (1+1)-dimensional integrable chains constructed above. Some families of functions F described above linearly depend on two parameters. Denote these parameters by γ_1, γ_2. The corresponding integrable chain
$$u_{k,t} = \gamma_1(\phi_{k,1}u_{1,x} + \cdots + \phi_{k,k+1}u_{k+1,x}) + \gamma_2(\psi_{k,1}u_{1,x} + \cdots + \psi_{k,k+1}u_{k+1,x})$$
is also linear in γ_1, γ_2. We claim that the following (2+1)-dimensional chain
$$u_{k,t} = (\phi_{k,1}u_{1,x} + \cdots + \phi_{k,k+1}u_{k+1,x}) + (\psi_{k,1}u_{1,y} + \cdots + \psi_{k,k+1}u_{k+1,y}) \tag{6.43}$$
is integrable from the viewpoint of the method of hydrodynamic reductions. For each case the reductions can be easily described.

For example, in the generic case
$$F(p) = \gamma_1 e^{u_2+u_1}(p-1) + \gamma_2 e^{u_2-u_1}(p+1)$$
formula (6.43) yields (2+1)-dimensional chain
$$u_{k,t} = e^{u_2+u_1}(u_{k+1,x} - u_{k,x}) + e^{u_2-u_1}(u_{k+1,y} + u_{k,y}), \qquad k = 1,2,3,... \tag{6.44}$$

After a change of variables of the form

$$x \to -\frac{1}{2}x, \quad y \to \frac{1}{2}y, \qquad u_1 \to \frac{1}{2}u_0, \quad u_2 \to u_1 + \frac{1}{2}u_0, \quad u_3 \to -2u_2 + \frac{1}{2}u_0, ...$$

(6.44) can be written as

$$u_{0,t} = e^{u_1}u_{0,y} + e^{u_1}(u_{1,y} - e^{u_0}u_{1,x}), \qquad u_{i,t} = e^{u_0+u_1}u_{i,x} + e^{u_1}(e^{u_0}u_{i+1,x} - u_{i+1,y}), \tag{6.45}$$

where $i = 1, 2, ...$ Probably (6.45) is a first example of a (2+1)-dimensional chain integrable from the viewpoint of the hydrodynamic reduction approach.

Triangular GT-systems related to integrable (2+1)-dimensional chains with fields $u_0, u_1, u_2, ...$ have the form

$$\begin{gathered}
\partial_i p_j = f_1(p_i, q_i, p_j, q_j, u_0, ..., u_n)\partial_i u_0, \qquad \partial_i q_j = f_2(p_i, q_i, p_j, q_j, u_0, ..., u_n)\partial_i u_0, \\
\partial_i\partial_j u_0 = h(p_i, q_i, p_j, q_j, u_0, ..., u_n)\partial_i u_0 \partial_j u_0, \\
\partial_i u_k = g_k(p_i, q_i, u_0, ..., u_{k+1})\partial_i u_0, \qquad k = 0, 1, 2, ...
\end{gathered} \tag{6.46}$$

Here $i \neq j,\ i, j = 1, ..., 3,\ p_1, ..., p_3,\ q_1, ..., q_3,\ u_0, u_1, u_2, ...,$ are functions of r^1, r^2, r^3. In particular, the GT-system associated with (6.45) has the form:

$$\partial_i p_j = \partial_i\partial_j u_0 = 0, \qquad \partial_i q_j = \Big(\frac{p_i q_i - p_j q_j}{p_i - p_j} - q_i q_j\Big)\partial_i u_0, \qquad \partial_i u_k = -\frac{p_i}{(p_i - 1)^k}\partial_i u_0.$$

The hydrodynamic reductions of (6.45) is given by the pair of semi-hamiltonian (1+1)-dimensional systems

$$r^i_y = e^{u_0}\Big(1 - \frac{1}{q_i}\Big)r^i_x, \qquad r^i_t = e^{u_0+u_1}\Big(\frac{1}{(p_i - 1)q_i} + 1\Big)r^i_x.$$

Chain (6.45) is the first member of an infinite hierarchy of pairwise commuting flows where the corresponding "times" are $t_1 = t,\ t_2,\ t_3, ...$ These flows and their hydrodynamic reductions can be described in terms of the generating function $U(z) = u_1 + u_2 z + u_3 z^2 + ...$ The hierarchy is given by

$$D(z)u_0 = e^{U(z)}\Big(u_{0,y} + U(z)_y - e^{u_0}U(z)_x\Big),$$

$$D(z_1)U(z_2) = e^{u_0+U(z_1)}U(z_2)_x + (1 + z_1)e^{U(z_1)}\Big(e^{u_0}\frac{U(z_1)_x - U(z_2)_x}{z_1 - z_2} - \frac{U(z_1)_y - U(z_2)_y}{z_1 - z_2}\Big),$$

where $D(z) = \frac{\partial}{\partial t_1} + z\frac{\partial}{\partial t_2} + z^2\frac{\partial}{\partial t_3} + ...$ The reductions can be written as

$$D(z)r^i = e^{u_0+U(z)}\Big(1 + \frac{1 + z}{(p_i - 1 - z)q_i}\Big)r^i_x.$$

Other (2+1)-dimensional integrable chains related to 2-dimensional vector spaces of solutions for system (6.39) are degenarations of (6.45). In particular $F = \gamma_1 e^{u_1}p + \gamma_2(p + u_2)$ leads to the following (2+1)-dimensional integrable generalization of (6.44):

$$u_{k,t} = e^{u_1}u_{k+1,x} + u_{k+1,y} + u_2 u_{k,y}, \qquad k = 1, 2, 3, ...$$

Conjecture. Any chain of the form (6.43) integrable by the hydrodynamic reduction method is a degeneration of (6.45).

We are planning to consider the problem of classification of integrable chains (6.43) in a separate paper.

7 Infinitesimal symmetries of triangular GT-systems

A scientific way to construct the functions $g_3, g_4, ...$ for different cases from Proposition 1 is related to infinitesimal symmetries of the corresponding GT-system[1]. The whole Lie algebra of symmetries is one the most important algebraic structures related to any triangular GT-system (4.25). In particular, this algebra acts on the hierarchy of the commuting flows for the corresponding chain (1.1).

A vector field

$$S=\sum_{j=1}^{N} X(p_j,u_1,...,u_s)\frac{\partial}{\partial p_j}+\sum_{m=1}^{\infty} Y_m(u_1,...,u_{k_m})\frac{\partial}{\partial u_m}, \qquad \frac{\partial Y_m}{\partial u_{k_m}}\neq 0 \tag{7.47}$$

is called a *symmetry* of the triangular GT-system (4.25) if it commutes with all ∂_i. Notice that it follows from the definition that

$$S(\partial_i u_1)=\partial_i(Y_1).$$

We call (7.47) a symmetry of shift d if $k_m = m+d$ for $m >> 0$. Let M be the minimal integer such that $k_m = m+d, m > M$. If the functions $g_i, i = 1, ..., M+d$ from (4.25) are known, then the functions $X, Y_1, ...Y_M$ can be found from the compatibility conditions

$$S(\partial_i p_j)=\partial_i S(p_j), \qquad S(\partial_i u_k)=\partial_i S(u_k), \quad k=1,...,M.$$

The functions $Y_{M+1}, Y_{M+2}, ...$ can be chosen arbitrarily. After that $g_{M+d+1}, g_{M+d+2}, ...$ are uniquely defined by the remaining compatibility conditions.

The generic case 1, $\mathbf{a_1}$. Looking for symmetries of shift one, we find $X = Y_1 = 0$ and $M = 1$. Hence without loss of generality we can take

$$S=\sum_{m=2}^{\infty} u_{m+1}\frac{\partial}{\partial u_m}$$

for the symmetry. This fact gives us a way to construct all functions $g_i, i > 3$ in the infinite triangular extension for the case 1, a_1. Indeed, it follows from the commutativity conditions $S(\partial_i u_k) = \partial_i S(u_k)$ that $g_{k+1} = S(g_k)$, where $k = 2, 3, ...$ In particular,

$$g_3=\frac{(p_j-u_1)(2p_ju_2-p_j-u_2^2)u_3}{u_1(u_1-1)(p_j-u_2)^2}.$$

[1]Note that these functions are not unique because of the triangular group of symmetries (1.2) acting on the fields $u_3, u_4, ...$

The functions g_i thus constructed are not linear in u_3. The corresponding chain (1.1) is equivalent to the chain constructed in Section 5 but not so simple.

It would be interesting to describe the Lie algebra of all symmetries in this case. Here we present the essential part for symmetry of shift 2:

$$X = \frac{p_j(p_j-1)u_3^2}{(p_j-u_2)u_2(u_2-1)}, \qquad Y_1 = \frac{u_1(u_1-1)u_3^2}{(u_1-u_2)u_2(u_2-1)},$$

$$Y_2 = -\frac{3}{2}u_4 + \frac{(2u_1-1)u_3^2}{u_2(u_2-1)} + u_3. \qquad \square$$

The case 1, $\mathbf{d_1}$. One can add fields $u_3, \ldots$ in such a way that the whole triangular GT-system admits the following symmetry of shift 1:

$$S = \frac{u_2}{u_1(u_1-1)} \sum_{i=1}^{N} p_i(p_i-1)\frac{\partial}{\partial p_i} + \sum_{i=1}^{\infty} u_{i+1}\frac{\partial}{\partial u_i}.$$

As in the previous example, one can easily recover the whole GT-system. For example,

$$\partial_i u_3 = \left(\frac{u_3(p_i+u_1-1)}{u_1(u_1-1)} + \frac{2u_2^2 p_i(p_i-1)}{u_1^2(u_1-1)^2}\right)\partial_i u_1. \qquad \square$$

Below we describe the symmetry algebra for the case 5, c_2 (in particular, for the Benney chain).

The case 5, $\mathbf{c_2}$. For the triangular GT-system (1.7), (1.8) there exists an infinite Lie algebra of symmetries $S_i, i \in \mathbb{Z}$, where S_i is a symmetry of shift i. The simplest symmetries are the following:

$$S_{-2} = \frac{\partial}{\partial u_1} + \sum_{i=3}^{\infty}\Big(- u_{i-2} + \sum_{k+m=i-3} u_k u_m - \sum_{k+m+l=i-4} u_k u_m u_l + \cdots \Big)\frac{\partial}{\partial u_i},$$

$$S_{-1} = \sum_{j=1}^{N} \frac{\partial}{\partial p_j} + \sum_{i=1}^{\infty}(i-1)u_{i-1}\frac{\partial}{\partial u_i},$$

$$S_0 = \sum_{j=1}^{N} p_j\frac{\partial}{\partial p_j} + \sum_{i=1}^{\infty}(i+1)u_i\frac{\partial}{\partial u_i},$$

$$S_1 = \sum_{j=1}^{N}(p_j^2+3u_1)\frac{\partial}{\partial p_j} + \sum_{i=1}^{\infty}(i+3)u_{i+1}\frac{\partial}{\partial u_i} + \sum_{i=2}^{\infty}\sum_{k+m=i} u_k u_m\frac{\partial}{\partial u_i} + \sum_{i=2}^{\infty} 3(i-1)u_1u_{i-1}\frac{\partial}{\partial u_i},$$

$$S_2 = \sum_{j=1}^{N}(p_j^3+4u_1p_j+5u_2)\frac{\partial}{\partial p_j} + \sum_{i=1}^{\infty}(i+5)u_{i+2}\frac{\partial}{\partial u_i} + \sum_{i=1}^{\infty} 4iu_1u_i\frac{\partial}{\partial u_i} + \sum_{i=2}^{\infty} 5(i-1)u_2u_{i-1}\frac{\partial}{\partial u_i} +$$

$$\sum_{i=1}^{\infty}\sum_{k+m=i+1}3u_ku_m\frac{\partial}{\partial u_i}+\sum_{i=3}^{\infty}\sum_{k+m+l=i}u_ku_mu_l\frac{\partial}{\partial u_i}.$$

The whole algebra is generated by S_1, S_2, S_{-1}, S_{-2}. It is isomorphic to the Virasoro algebra with zero central charge.

Let D_{t_i} be the vector fields corresponding to commuting flows for the Benney chain. Here $D_{t_1} = D_x$, $D_{t_2} = D_t$. Then the commutator relations

$$[S_1, D_{t_i}] = (i+1)D_{t_{i+1}}$$

hold. Thus the vector field S_1 plays the role of a master-symmetry for the Benney hierarchy. □

The case 6. In this case there exist infinitesimal symmetries of form

$$T_i = u_{i+1}\frac{\partial}{\partial u_1}+u_{i+2}\frac{\partial}{\partial u_2}+...,\ i=0,1,2,...$$

$$S_i=\sum_{j=1}^{N}p_j^{i+1}\frac{\partial}{\partial p_j}+u_{i+2}\frac{\partial}{\partial u_2}+2u_{i+3}\frac{\partial}{\partial u_3}+3u_{i+4}\frac{\partial}{\partial u_4}+...,\ i=-1,0,1,2,...$$

Note that $[S_i, S_j] = (j-i)S_{i+j},\ [T_i, T_j] = 0,\ [S_i, T_j] = jT_{i+j}$. □

References

[1] *E.V. Ferapontov, K.R. Khusnutdinova*, On integrability of (2+1)-dimensional quasilinear systems, Comm. Math. Phys. **248** (2004) 187-206,

[2] *E.V. Ferapontov, K.R. Khusnutdinova*, The characterization of 2-component (2+1)-dimensional integrable systems of hydrodynamic type, J. Phys. A: Math. Gen. **37**(8) (2004) 2949 - 2963.

[3] *E.V. Ferapontov, K.R. Khusnutdinova*, Hydrodynamic reductions of multidimensional dispersionless PDEs: the test for integrability, J. Math. Phys. **45**(6) (2004) 2365 - 2377.

[4] *M.V. Pavlov*, Classification of the Egorov hydrodynamic chains. Theor. Math. Phys. **138** No. 1 (2004) 55-71.

[5] *M.V. Pavlov*, Classification of integrable hydrodynamic chains and generating functions of conservation laws, J. Phys. A: Math. Gen. **39**(34) (2006) 10803–10819.

[6] *E.V. Ferapontov, D.G. Marshal*, Differential-geometric approach to the integrability of hydrodynamic chains: the Haanties tensor, Math. Ann. **339**(1), (2007) 61–99.

[7] *D.J. Benney*, Some properties of long nonlinear waves, Stud. Appl. Math. **52** (1973) 45-50.

[8] *B.A. Kupershmidt, Yu.I. Manin*, Long waves equation with free surface. I. Conservation laws and solutions. Func. Anal. and Appl., **11**(3) (1977) 31-42.

[9] *V.E. Zakharov*, On the Benney's Equations, Physica 3D (1981) 193-200.

[10] *J. Gibbons, S.P. Tsarev*, Reductions of Benney's equations, Phys. Lett. A, **211** (1996) 19-24.

[11] *A.V. Odesskii, V.V. Sokolov*, Systems of Gibbons-Tsarev type and integrable 3-dimensional models, arXiv:0906.3509

[12] *A.V. Odesskii, V.V. Sokolov*, Integrable (2+1)-dimensional hydrodynamic type systems, Theor. and Math. Phys, to be published.

[13] *A.V. Odesskii, V.V. Sokolov*, Integrable pseudopotentials related to generalized hypergeometric functions, arXiv:0803.0086

[14] *I.M. Gelfand, M.I. Graev, V.S. Retakh*, General hypergeometric systems of equations and series of hypergeometric type, Russian Math. Surveys 47 (1992), no. 4, 1–88

[15] *A.V. Odesskii, V.V. Sokolov*, Integrable pseudopotentials related to elliptic curves, Teoret. and Mat. Fiz., **161**(1) (2009) 21–36, arXiv:0810.3879

[16] *B.A. Kupershmidt*, Deformations of integrable systems, Proc. Roy. Irish Acad. Sect. A, **83**(1) (1983) 45-74.

[17] *L. Martinez Alonso, A.B. Shabat*, Hydrodynamic reductions and solutions of a universal hierarchy , Teoret. and Mat. Fiz., **140**(2) (2004) 1073–1085